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AN UPPER BOUND FOR THE CARDINALITY OF AN s-DISTANCE SUBSET IN REAL EUCLIDEAN SPACE

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If X is an s-distance subset in $\mathbf{R}^d$, then $|X| < \binom{d+s}{s}+\binom{d+s-1}{s-1}$.

A subset X in a metric space M is called an s-distance subset in M if there are s distinct distances $\alpha_1, \alpha_2, \ldots, \alpha_s$, and all the α_i are realized. Delsarte—Goethals—Seidel [2] have shown that the cardinality $|X|$ of an s-distance subset X in the unit sphere $S^d=\{(x_1, \ldots, x_{d+1})\,|\,x_1^2+\ldots+x_{d+1}^2=1\}\subset \mathbf{R}^{d+1}$ is bounded from above as

$$|X| \leqq \binom{d+s}{s}+\binom{d+s-1}{s-1}. \tag{1}$$

Larman—Rogers—Seidel [3] have shown that $|X|\leqq(d+1)(d+4)/2$ for a 2-distance subset in $\mathbf{R}^d$. Here we first prove the following

Theorem 1. *If X is an s-distance subset in $\mathbf{R}^d$, then*

$$|X| \leqq \binom{d+s}{s}+\binom{d+s-1}{s-1}. \tag{2}$$

(We remark that this upper bound is the same as the upper bound (1) for S^d, and that this is the same as the Larman—Rogers—Seidel bound $(d+1)(d+4)/2$ if $s=2$.)

Next we prove the following

Theorem 2. *The equality does not hold in (2) in Theorem 1.*

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1. Proof of Theorem 1

We give two different proofs. The first one is based on the argument by Larman—Rogers—Seidel [3]. The second one is more geometric and makes use of the upper bound (1) for S^d.

First Argument. We prepare with the following.

Lemma. *Let $t_1, t_2, \ldots, t_d$ be independent variables, and let $t_0 = t_1^2 + t_2^2 + \ldots + t_d^2$. Let W_s be the space spanned by the monomials $t_0^{i_0} t_1^{i_1} \ldots t_d^{i_d}$ with $i_0 + i_1 + \ldots + i_d \leqq s$. Let h_s be the dimension of W_s. Then*

$$h_s \leqq \binom{d+s}{s} + \binom{d+s-1}{s-1}. \tag{3}$$

Proof of Lemma. We use induction on s. The assertion is trivially true for $s=1$. We assume the assertion is true for $s-1$, namely

$$h_{s-1} \leqq \binom{d+s-1}{s-1} + \binom{d+s-2}{s-2}.$$

Let Harm(j) be the space of harmonic polynomials in $t_1, t_2, \ldots, t_d$. Then

$$\textit{dimension of } \operatorname{Harm}(j) = \binom{d+j-1}{j} - \binom{d+j-3}{j-2}$$

and so

$$\sum_{j=0}^{s} (\textit{dimension of } \operatorname{Harm}(j)) = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}.$$

(Cf. [2, Theorem 3.2].)
Let f be any polynomial of degree i in $t_1, t_2, \ldots, t_d$. Then f is uniquely decomposed as

$$f = f_i + t_0 f_{i-2} + t_0^2 f_{i-4} + \ldots + t_0^{\left[\frac{i}{2}\right]} f_{i-2\left[\frac{i}{2}\right]}$$

where $f_j \in \operatorname{Harm}(j)$. (Cf. [2, Theorem 3.1].) A monomial $t_0^{i_0} t_1^{i_1} \ldots t_d^{i_d}$ in W_s is contained in W_{s-1} if $f_i = 0$ for the polynomial $f = t_0^{i_0} t_2^{i_2} \ldots t_d^{i_d}$ of degree $i_1 + i_2 + \ldots + i_d = i$. Therefore there are at most

$$\sum_{i=0}^{s} (\textit{dimension of } \operatorname{Harm}(i))$$

linearly independent elements in W_s/W_{s-1}. Therefore

$$\begin{aligned} h_s &\leqq h_{s-1} + \binom{d+s-1}{s} + \binom{d+s-2}{s-1} \\ &\leqq \binom{d+s}{s} + \binom{d+s-1}{s-1}. \quad \blacksquare \end{aligned}$$

(We remark that in fact equality holds in (3).)

We start the proof of Theorem 1.

Let $\vec{s}=(s_1, \ldots, s_d)$ and $\vec{t}=(t_1, \ldots, t_d)$ be two elements in $\mathbf{R}^d$. Let us define $F_{\vec{s}}(\vec{t})$ by

$$F_{\vec{s}}(\vec{t}) = \prod_{j=1}^{s} (\|\vec{t}-\vec{s}\|^2-\alpha_j^2)$$

$$= \prod_{j=1}^{s} \left(\|\vec{t}\|^2-2\sum_{i=1}^{d} s_i t_i+\|\vec{s}\|^2-\alpha_j^2\right).$$

For a fixed $\vec{s}$, the function $F_{\vec{s}}(\vec{t})$ is a linear combination of monomials $(\|\vec{t}\|^2)^{i_0} t_1^{i_1} t_2^{i_2} \ldots t_d^{i_d}$ with $i_0+i_1+\ldots+i_d \leqq s$, hence $F_{\vec{s}}(\vec{t}) \in W_s$. Since

$$F_{\vec{s}}(\vec{t}) = 0 \quad \textit{if} \quad \vec{t} \in X \quad \textit{and} \quad \vec{t} \neq \vec{s}$$

and

$$F_{\vec{s}}(\vec{t}) = (-1)^s \prod_{j=0}^{s} \alpha_j^2 \quad \text{if} \quad \vec{t} \in X \quad \text{and} \quad \vec{t} = \vec{s},$$

the functions $F_{\vec{s}}$ (with $\vec{s} \in X$) are linearly independent. Therefore

$$|X| \leqq \text{dimension of } W_s = h_s \leqq \binom{d+s}{s} + \binom{d+s-1}{s-1}. \quad \blacksquare$$

We remark that this bound is also true for s-near-distance subsets in $\mathbf{R}^d$. Here an s-near-distance subset means that each distance is "sufficiently" close to one of $\alpha_1, \alpha_2, \ldots, \alpha_s$. (cf. [3], Remark, page 262.)

Second Argument. First we remark that the argument in [2, Theorem 4.8] which proves the upper bound (1) for the cardinality of an s-distance subset in S^d is also true for an s-near-distance subset in S^d. Let X be an s-distance subset in $\mathbf{R}^d$. By applying a similarity transformation, we may assume without loss of generality that X is contained in a small neighborhood V_1 of the origin 0 in $\mathbf{R}^d$. By a natural or stereographic projection, we can map the neighborhood V_1 onto a neighborhood V_2 of the south pole P in $S^d=\{(x_1, \ldots, x_{d+1})\,|\,x_1^2+\ldots+x_d^2+(x_{d+1}-1)^2=1\}$. Then the s-distance subset X in $\mathbf{R}^d$ is mapped onto an s-near-distance subset (in the sense defined above) in S^d if the neighborhood V_1 is small enough. Hence the remark mentioned at the beginning of this second argument completes the proof. We note that this argument also gives the same upper bound for s-near-distance subsets in $\mathbf{R}^d$.

2. Proof of Theorem 2

Here we give a sketch of the proof of Theorem 2. The details will be left to the reader. We assume that the reader is familiar with the paper [2] by Delsarte—Goethals—Seidel.

In [2, Theorem 6.6] it is shown that, if X is an s-distance subset in S^d ($=\Omega_{d+1}$ in the notation of [2]) and $|X|$ attains the bound (1), then the set $A(X)= =\{\theta_i=\cos\alpha_i\,|\,i=1, 2, \ldots, s\}$, the set of inner products (x, y) for $x, y \in X$ and $x \neq y$, must be equal to the set of the zeros of the following Jacobi polynomial of degree s:

$$P_s^{\left(\frac{d+1}{2}, \frac{d-1}{2}\right)}(x) = R_s(x) := Q_0(x)+Q_1(x)+\ldots+Q_s(x),$$

where the $Q_i(x)$ are the Gegenbauer polynomials of degree i with $Q_i(1)=$ *the dimension of* Harm (i) *in* $\mathbf{R}^{d+1}$.

For an s-near-distance set X in S^d let $A(X)=\{\theta_1, \theta_2, \ldots, \theta_s\}$ be the associated inner-product set, namely for each (x, y) with $x, y \in X$ and $x \neq y$ there exists some θ_i such that $\|(x, y)-\theta_i\| < \varepsilon$ for sufficiently small ε. Then a careful examination of the proof of Delsarte—Goethals—Seidel [2] shows that the same conclusion as above is valid for an s-near-distance subset X in S^d for which $|X|$ attains the bound (1). (Namely, the set $A(X)$ approaches the set of the zeros of $R_s(x)$ as $\varepsilon \to 0$.) This implies that the distribution of the set $A(X)$ is essentially the same as that of the zeros of the Jacobi polynomial $R_s(x)$, and this implies in turn that $A(X)$ is distributed fairly homogeneously in the interval $[-1, 1]$, and in particular that $A(X)$ is not contained in a small neighborhood around 1.

On the other hand, the second argument in the proof of Theorem 1 shows that if there is an s-distance subset X in $\mathbf{R}^d$ then we can construct an s-near-distance subset X with the same cardinality in an arbitrarily small neighborhood of S^d. This implies that $A(X)$ is contained in a small neighborhood around 1. Therefore we have a contradiction if $|X|$ attains the bound (1). Thus we complete the proof of Theorem 2. ∎

3. Remarks

(i) It is known that the upper bound (1) for s-distance subsets in S^d is not attained for $s \geqq 3$ (by Bannai—Damerell [1] when combined with [2, Theorem 6.6]), while there are some examples which attain the bound (1) for S^d for $s \leqq 2$ (see [2]).

(ii) It would be interesting to know how much the bounds (1) and (2) can be improved. However the right magnitude for the bounds (for both S^d and R^d) is $(d^s/s!)(1+o(1))$ if $d \to +\infty$, because there exist trivial $s-(v, s, 1)$ Steiner systems (namely all s-subsets of a v-set). We remark that the bounds (1) and (2) are already of the same magnitude as $(d^s/s!)(1+o(1))$.

References

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